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Spectral Representation of Semistable Processes, and Semistable Laws on Banach Spaces

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We introduce the notion of semistable processes and semistable random measures; and give a characterization of semistable laws on Banach spaces. Using this charcterization, we discuss the existence of semistable random measures, define the stochastic integrals with respect to these measures, and obtain the spectral representations of arbitrary (not necessairly symmetric) semistable and stable processes. In addition, we give a criterion of independence for stochastic integrals.

1. INTRODUCTION

Motivated from the well-established fact that the spectral representation of a Gaussian stationary process X, in terms of the Fourier transform of an independent increment Gaussian process, played an important role in the statistical and probabilistic study of X, Schilder [16] obtained similar representation for (finite) symmetric stable process of any index, $0<\alpha<2$. This representation was later extended by Kuelbs [7] for an arbitrary (not necessarily finite) symmetric stable process, which is continuous in probability. This representation for, $1<\alpha<2$, was also obtained independently by Bretagnolle *et al.* [3]. The main purpose of this paper is to enlarge the scope of these spectral representations to a larger class of

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processes including those studied in [3, 7, 16]: We introduce the class of r-semistable index α processes and obtain several results, including spectral representation, for these processes. We expect that the results presented here will be useful in studying sample path properties as well as the prediction and estimation questions for this class of processes. In the following, we summarize our results more precisely.

We believe that we make two main contributions in this paper: First, we give a definition of stochastic integrals for a suitable class of functions relative to semistable and stable random measures (Theorem 4.1); second, we obtain spectral representations for arbitrary (not necessarily symmetric) semistable and stable index, $0<\alpha<2$, $\alpha \neq 1$ processes X, which are continuous in probability (Theorem 5.1). (If X is symmetric, it is also shown that these representations hold for all, $0<\alpha<2$, including $\alpha=1$.) Crucial to our proof of the spectral representation results is a characterization of a semistable index α probability measure on a separable Banach space B in terms of a finite measure on a suitable ring of B (Theorem 3.1); this measure on the ring plays a similar role as did the spectral measure (on the unit sphere of B) of the symmetric stable measure in obtaining the spectral representation of a symmetric stable process in [3, 7, 16]. This characterization is similar to that of Karkowiak [5], but the form of the characteristic (ch.) function obtained here is slightly different and more suitable for the purpose of defining semistable stochastic integrals. Besides these results, we obtain a condition, for independence of infinitely divisible Banach-valued random variables, in terms of their Lévy measure (Theorem 6.1); using this, we give a criterion for independence of stochastic integrals (Corollary 6.3).

The organization of the rest of the paper is as follows: Section 2 contains preliminaries; Section 3 contains the characterization of semistable probability measures on Banach spaces. Sections 4 and 5 contain, respectively, the stochastic integrals relative to stable and semistable random measures, and spectral representation results for semistable and stable processes. Section 6 contains results dealing with the independence of infinitely divisible random variables and stochastic integrals.

2. Preliminaries

In this section, we recall some definitions and known facts; we also record some notation and conventions, which we shall use throughout the paper.

Let $0<r<1$ and let B be a real separable Banach space. A Borel probability (p.) measure μ on B is called *r-semistable* if there exist $x_n \in B$,

$a_n > 0$, a sequence $\{k_n\}$ of positive integers and a Borel p. measure ν on B such that $k_n k_{n+1}^{-1} \to r$ and

$$a_n \cdot \nu^{*k_n} * \delta_{x_n} \xrightarrow{w} \mu, \tag{2.1}$$

as $n \to \infty$, where $*$ and $\to^w$ denote, respectively, the convolution of measures and the usual weak convergence of measures; and, for a given number $a \neq 0$ and a measure ρ, $a \cdot \rho = \rho \circ \tau_a^{-1}$,where $\tau_a(x) = ax$. (If $a = -1$, then $a \cdot \rho$ will be denoted by $\bar{\rho}$.) It is shown in [4] that μ is r-semistable if and only if μ is infinitely divisible (i.d.) and

$$\mu^{r^n} = r^{n/\alpha} \cdot \mu * \delta_{x_n}, \tag{2.2}$$

for every $n = \pm 1, \pm 2, \ldots$, where μ^t denotes the tth power of μ (the existence and properties of which are also proved in [4]), $0 < \alpha < 2$ and is uniquely determined by μ; α is referred to as *the index* of μ. If x_n in (2.1) or equivalently in (2.2) is equal to 0, for every n, then μ is called *strictly r-semistable*. We refer the readers to [4] for the above and another characterization of r-semistable p. measures on locally convex spaces.

For a given $0 < r < 1$ and $0 < \alpha < 2$, throughout, we shall use the notation r-SS(α) (resp. $S(\alpha)$) for the phrase "r-semistable index α" (resp. for "stable index α"). If X is a random variable (r.v.) with values in a separable Banach space B, then $\mathscr{L}_X$ will denote the law of X in B. Now we are ready to define a semistable process.

Let, $0 < r < 1$, $0 < \alpha < 2$, and let $X = \{X_\lambda : \lambda \in \Lambda\}$ be a stochastic process with values in a separable Banach space B (Λ an arbitrary index set). Then X will be called an r-SS(α) (resp. a *strictly* r-SS(α)) stochastic process, if for any $\lambda_1, \ldots, \lambda_n \in \Lambda$, the law $\mathscr{L}_{(X_{\lambda_1}, \ldots, X_{\lambda_n})}$ on B^n is an r-SS(α) (resp. a strictly r-SS(α)) p. measure. Using result of [4], it is easy to show that if $\alpha \neq 1$, then an r-SS(α) process $X = \{X_\lambda : \lambda \in \Lambda\}$ can be written as $X_\lambda = \theta(\lambda) + Y_\lambda$, where $\{Y_\lambda : \lambda \in \Lambda\}$ is a strictly r-SS(α) process and θ is a deterministic function. We refer to θ as the *centering function* of X. It follows, from Theorem 3.1, that if $\alpha \neq 1$ or if X is symmetric then X is an r-SS(α) stochastic process if and only if $\mathscr{L}_{(\sum_j^n a_j X_{\lambda_j})}$ is an r-SS(α) p. measure on B, for every choice $\lambda_1, \ldots, \lambda_n$ in Λ and $a_1, \ldots, a_n$ in R, the reals.

Finally, we record some more notation and conventions that will remain fixed throughout the paper: By a measure on a topological space, we will always mean that it is defined on its Borel sets. If B is a Banach space, then B^* and $\langle , \rangle$ will, respectively, denote the topological dual of B and the natural duality between B and B^*. In a Banach space B with norm $\|\cdot\|$, Δ_n and Δ will, respectively, denote the sets $\{x \in B: r^{(n+1)/\alpha} < \|x\| \leqslant r^{n/\alpha}\}$, for $n = 0, \pm 1, \pm 2, \ldots$, and $\{x \in B: \|x\| = 1\}$. If μ is a p. measure on a Banach space, then $\hat{\mu}$ will denote the ch. function of μ.

3. A Characterization of Semistable p. Measures on Banach Spaces

Semistable r.v.'s on real line were introduced by P. Lèvy [8]. Two characterizations of r-SS(α) p. measures on general locally convex spaces, as mentioned in the Introduction, are obtained in [4] (see also [6]). In this section, we give another characterization of an r-SS(α) p. measure μ on a separable Banach space in terms of a finite measure Γ on Δ_0, and obtain the characteristic function $\hat{\mu}$ of μ in a more explicit form. This particular form of $\hat{\mu}$, as we will see in Sections 4 and 5, played an important role while defining the stochastic integral and obtaining the spectral representation for r-SS(α) processes. As pointed out in Section 1, this characterization is similar to that given in [5], but the form of the ch. function of the r-SS(α) p. measure is obtained in Theorem 3.1 is different in that it is obtained as a function of $|\langle x, y\rangle|^\alpha$ and another function k_α. This form and the properties of k_α play important role in proving the crucial inequality (4.11) as well as in the other analysis in the rest of the paper. It is for this reason, we include Theorem 3.1.

Before we state and prove the main result of this section, we make a few remarks regarding functions k_α, $\bar{k}_\alpha$ which are defined below. These functions arise naturally in the description of ch. functions of r-SS(α) p. measures and are, in fact, important for us throughout the paper.

Let $0<r<1$ and $0<\alpha<2$; set

$$k'_\alpha(t)=\begin{Bmatrix} e^{-\alpha t}\sum_n r^{-n}\{1-\exp(ir^{n/\alpha}e^t)\} & \text{if } 0<\alpha<1, \\ e^{-\alpha t}\sum_n r^{-n}\{1+ir^{n/\alpha}e^t-\exp(ir^{n/\alpha}e^t)\} & \text{if } 1<\alpha<2 \end{Bmatrix},$$

$$\bar{k}'_\alpha(t)= =e^{-\alpha t}\sum_n r^{-n}\{1-\cos(r^{n/\alpha}e^t)\}, \qquad \text{if } 0<\alpha<2.$$

Then, for $t\neq 0$, we define

$$k_\alpha(t)=\mathscr{R}ek'_\alpha(\log|t|)+i\operatorname{sgn}(t)\,\mathscr{I}mk'_\alpha(\log|t|) \tag{3.1}$$

if $0<\alpha<2$, $\alpha\neq 1$; and

$$\bar{k}_\alpha(t)=\bar{k}'_\alpha(\log|t|), \qquad \alpha\in(0,2). \tag{3.2}$$

Here and throughout the paper $\sum_n$ will stand for $\sum_{n=-\infty}^{n=+\infty}$. We note that if $\alpha\neq 1$, $\bar{k}_\alpha(e^t)=\mathscr{R}ek'_\alpha(t)=\mathscr{R}ek_\alpha(e^t)$, and that the series defining the functions k'_α, $\bar{k}'_\alpha$ are uniformly convergent on bounded intervals; and hence both functions k'_α, $\bar{k}'_\alpha$ are continuous on R. Further, both k'_α and $\bar{k}'_\alpha$ are periodic with period $-\alpha^{-1}\log r$; and we have the following inequalities:

$$0<c_0\equiv\inf_{t\in R\setminus\{0\}}\mathscr{R}ek_\alpha(t)\leqslant\sup_{t\in R\setminus\{0\}}|k_\alpha(t)|\equiv c_1<\infty, \tag{3.3}$$

for $\alpha \in (0, 2)$, $\alpha \neq 1$, and, for all $\alpha \in (0, 2)$,

$$0 < c_0 = \inf_{t \in R \setminus \{0\}} \bar{k}_\alpha(t) \leqslant \sup_{t \in R \setminus \{0\}} \bar{k}_\alpha(t) \leqslant c_1 < \infty. \tag{3.4}$$

THEOREM 3.1. *Let μ be an r-SS(α) p. measure, $0 < \alpha < 2$, $\alpha \neq 1$, on a Banach space B with Lévy measure F; then there exist an x_0 in B and a strictly r-SS(α) p. measure μ_c such that $\mu = \delta_{x_0} * \mu_c$ with*

$$\hat{\mu}_c(y) = \exp - \left\{ \int_{\Delta_0} |\langle x, y \rangle|^\alpha k_\alpha(\langle x, y \rangle)\, d\Gamma \right\}, \tag{3.5}$$

where k_α is as in (3.1) and $\Gamma = F/\Delta_0$ (here and in the following $|t|^\alpha k_\alpha(t)$ is interpreted as equal to zero at $t = 0$). If μ is symmetric, then for any $\alpha \in (0, 2)$, $\hat{\mu}(y)$ is given by the right side of Eq. (3 5) with k_α replaced by $\bar{k}_\alpha$ (see 3.2).

Conversely, if Γ is any measure on a Borel subset T of B satisfying

$$\int_T \|x\|^\alpha \, d\Gamma < \infty, \tag{3.6}$$

for a given $\alpha \in (0, 2)$; then if $0 < \alpha < 1$, $\hat{\mu}_c$, given by (3.5), with Δ_0 replaced by T, is the ch. function of an r-SS(α) p. measure; and if, in addition, B is of type 2 [1, p. 158], then the same result holds for $1 < \alpha < 2$. Further, if in (3.5) k_α is replaced by $\bar{k}_\alpha$ and Δ_0 by T, then $\hat{\mu}_c$ is the ch. function of a symmetric r-SS(α) p. measure for every $\alpha \in (0, 2)$, provided B is of type 2, when $\alpha \in [1, 2)$. Finally, if $T = \Delta_0$ (in which case (3.6) is equivalent to $\Gamma(\Delta_0) < \infty$), then the p. measure μ_c determines uniquely the measure Γ in the nonsymmetric case and only the measure $\Gamma + \bar{\Gamma}$ in the symmetric case. (The element x_0 and the measure μ_c appearing in the statement of the theorem are unique.) The element x_0 will be called the *centering element* of μ. The measure $\Gamma \equiv F/\Delta_0$ will be referred to as the *spectral measure* of μ and μ_c; if μ is symmetric Γ will be assumed symmetric).

Proof. We prove the direct part for $0 < \alpha < 1$; the proofs for $\alpha \in (1, 2)$ and for the symmetric cases are similar. According to Theorem 4 of [4, p. 215], we can write $\mu = \delta_{x_0} * \mu_c$, where $x_0 \in B$, μ_c is strictly r-SS(α) p. measure and

$$\log \hat{\mu}_c(y) = \int_B (e^{i\langle x, y \rangle} - 1)\, dF; \tag{3.7}$$

further, F satisfies the condition

$$r^{n/\alpha} \cdot F = r^n F, \tag{3.8}$$

for are all $n = \pm 1, \pm 2, \cdots$. From (3.7), (3.8), the dominated convergence theorem and the change of variable formula, one easily obtains (3.5).

For the converse part, let r and α be fixed and define the measure F on the Borel sets A of B by

$$F(A) = \sum_n r^n \Gamma(r^{n/\alpha} A \cap T). \tag{3.9}$$

Then, (3.6) implies and is infact equivalent to

$$F\{\|x\| > \delta\} < \infty \tag{3.10}$$

for every $\delta > 0$. This follows from (3.9) and simple inequalities. Now (3.8) and (3.10) imply (see, for example, [4]) that

$$\int_B \min(1, \|x\|^2)\, dF < \infty \tag{3.11}$$

and

$$\int_B \min(1, \|x\|)\, dF < \infty \qquad \text{if } 0 < \alpha < 1. \tag{3.12}$$

Now if $0 < \alpha < 1$, it follows, from (3.12) and Theorem 6.3 of [1, p. 138], that F is a Lèvy measure. Hence, using (3.8), it follows, from Theorem 4 of [4], that right side of (3.7) is the logarithm of the ch. function of an r-SS(α) p. measure. Hence, since it is easy to verify, using (3.1) and (3.9), that $\log \hat{\mu}_c(y) = -\int_T |\langle x, y\rangle|^\alpha k_\alpha(\langle x, y\rangle)\, d\Gamma$, it follows that $\hat{\mu}_c$ is the ch. function of an r-SS(α) p. measure. If $\alpha \in (1, 2)$, (3.11) implies, in type 2 Banach spaces, that F is a Lèvy measure (Theorem 7.6 of [1, p. 163]). Hence, using (3.8) and Theorem 4 of [4], it follows that $\hat{\mu}_c(y)$ given by

$$\log \hat{\mu}_c(y) = \int_B (e^{i\langle x, y\rangle} - 1 - i\langle x, y\rangle)\, dF,$$

and hence also given by (3.5) (use (3.1) and (3.9) again), with Δ_0 replaced by T, is the ch. function of an r-SS(α) p. measure. The proof in the symmetric case is similar and is so omitted. Finally, the last part of the theorem follows from the uniqueness of the Lèvy measure and the fact that $\Gamma = F/\Delta_0$.

Remark 3.2. (i) If $T \neq \Delta_0$, then the measure Γ, in the converse part of Theorem 3.1, is not uniquely determined by F. In fact, the measures $\Gamma_1 = \delta_{\{r^{1/\alpha}\}}$ and $\Gamma_2 = r\delta_{\{1\}}$ on the set $T = \{r^{1/\alpha}, 1\}$ yield the same Lèvy measure F on R through the formula (3.9).

(ii) The centering element x_0 is the mean of μ if $1<\alpha<2$ and satisfies the relation: $i\langle x_0, y\rangle = \lim_{s\to\infty} 1/s \log \hat{\mu}(sy)$, for all y, if $0<\alpha<1$.

Simple proofs of these are omitted.

We conclude this section by recording two propositions; these will be needed for the proof of Theorem 5.1. We included these here (rather than in Section 5) mainly because their proofs are related to the material presented in this section.

PROPOSITION 3.3. *Let $\alpha \in (0,2)$, $\alpha \neq 1$; and let $\mu, \mu_1, \mu_2, \ldots$ be r-SS(α)* p. *measures on R with centering elements $\theta, \theta_1, \theta_2, \ldots$. If $\mu_n \to^w \mu$, as $n\to\infty$ then $\theta_n \to \theta$, as $n\to\infty$.*

Proof. Since $\mu_n \to^w \mu$, we have $\mu_n * \bar{\mu}_n \to^w \mu * \bar{\mu}$; hence, using (3.5), it follows that $\{\int_{\Delta_0} |s|^\alpha \bar{k}_\alpha(s)\, d\Gamma_n\}$ is convergent, where Γ_n is the spectral measure of μ_n and $\bar{k}_\alpha$ is as in (3.2). Hence, since for all n,

$$\Gamma_n(\Delta_0) \leqslant (rc_0)^{-1} \int_{\Delta_0} |s|^\alpha \bar{k}_\alpha(s)\, d\Gamma_n,$$

where c_0 as in (3.4), $\{\Gamma_n(\Delta_0)\}$ is bounded. Hence $\{\Gamma_n\}$ has a weakly convergent subsequence, say $\{\Gamma_{n_k}\}$; but then $\{\theta_{n_k}\}$ must converge to θ by the uniqueness of the centering element. This implies $\theta_n \to \theta$.

PROPOSITION 3.4. *Let $0<r<1$ and $0<\alpha<2$. Let $\{X_n : n=1,2,\ldots\}$ be a strictly r-SS(α) process with $\sum_{n\geqslant 1} X_n^2 < \infty$* a.s., *then μ, the law of $\{X_n\}$ in l_2, is a strictly r-SS(α)* p. *measure.*

Proof. Let π_n is the natural projection of l_2 onto R^n, the n-Euclidean space. Then, since $\mu_n \equiv \mu\circ\pi_n^{-1} = \mathscr{L}_{(X_1,\ldots,X_n)}$, μ_n is strictly r-SS(α) p. measure on R^n. Hence μ_n is i.d., for every n; it follows that μ is i.d. [17]. Let $\{\mu^t : t>0\}$ be the associated semigroup of p. measures (recall μ^t is the tth power of μ). Now $\{\mu^t\circ\pi_n^{-1} : t>0\}$ is a semigroup of p. measures which is right continuous at 0 with $\mu^1\circ\pi_n^{-1} = \mu_n$; it follows from the uniqueness of $\{\mu_n^t : t>0\}$ that

$$\mu^t\circ\pi_n^{-1} = \mu_n^t$$

(see [4]). Thus since, for every n, $r^{1/\alpha}\cdot\mu_n = \mu_n^r$, we have $r^{1/\alpha}\cdot(\mu\circ\pi_n^{-1}) = \mu^r\circ\pi_n^{-1}$. Therefore, for every n, $(r^{1/\alpha}\cdot\mu)\circ\pi_n^{-1} = \mu^r\circ\pi_n^{-1}$, showing $r^{1/\alpha}\cdot\mu = \mu^r$. This completes the proof.

4. Construction of and Stochastic Integrals Relative to Semistable and Stable Random Measures

In this section, first we define and outline the construction of symmetric as well as nonsymmetric semistable and stable random measures; then we prove the central result of the section (Theorem 4.1), which gives the definition of stochastic integrals relative to these random measures.

Let $(T, \sigma(T))$ be a measurable space. A function $M: \sigma(T) \to L_0(\Omega, \mathscr{F}, P) \equiv L_0$, the vector space of real r.v.'s on the p. space $(\Omega, \mathscr{F}, P)$, is called an *independently scattered random measure* (or just *random measure*) if for every sequence $\{A_n\}$ of disjoint sets in $\sigma(T)$, the r.v.'s $M(A_n)$, $n = 1, 2, \ldots$, are independent and

$$M\left(\bigcup_{n \geqslant 1} A_n\right) = \sum_{n \geqslant 1} M(A_n), \tag{4.1}$$

where the series is assumed to converge in probability (and hence also a.s.).

Let now T be a Borel subset of R; and let m be a measure on $\mathscr{B}(T)$, the Borel subsets of T, satisfying

$$\int_T |s|^\alpha \, dm < \infty, \qquad 0 < \alpha < 2. \tag{4.2}$$

Let $0 < r < 1$, and $0 < \alpha < 2$, $\alpha \neq 1$. Then, a random measure M on $\mathscr{B}(T)$ is called an r-SS(α) *random measure with control measure m* if

$$\hat{\mathscr{L}}_{M(A)}(t) = \exp - \left\{ \int_A |ts|^\alpha \, k_\alpha(ts) \, dm \right\}, \tag{4.3}$$

for every $A \in \mathscr{B}(T)$ and $t \in R$, where k_α is as in (3.1). The measure M is called $S(\alpha)$ *random measure with control measure m* if

$$\hat{\mathscr{L}}_{M(A)}(t) = \exp - \left\{ \int_A |ts|^\alpha \, h_\alpha(ts) \, dm \right\}, \tag{4.4}$$

for every $A \in \mathscr{B}(T)$ and $t \in R$, where

$$h_\alpha(t) = 1 - i \operatorname{sgn}(t) \tan \frac{\pi\alpha}{2}. \tag{4.5}$$

Now let $\alpha \in (0, 2)$. If (4.3) holds with $k_\alpha(ts)$ replaced by $\bar{k}_\alpha(ts)$, then M is called *symmetric* r-SS(α) *random measure with control measure m*; if (4.4) holds with h_α replaced by 1 *then M is called symmetric* $S(\alpha)$ *random measure with control measure m.*

Remark 4.1. The existence of r-SS(α) and S(α) random measures follows from an application of Kolmogorov's existence theorem; this can also be proved using the classical work of Pre'kopa [13, 14].

Now we proceed to define stochastic integrals relative to r-SS(α) and S(α) random measures. First a few more notations: For a given measure space $(T, \sigma(T), \nu)$, $L_\alpha(T, \nu)$ will denote the usual Banach space $L_\alpha(T, \sigma(T), \nu)$, if $1 \leqslant \alpha < \infty$; and the usual linear topological metric space if $0 < \alpha < 1$.

Let $f = \sum_{j=1}^{n} a_j I_{A_j}$ be a simple function on $(T, \mathscr{B}(T))$ and let M be a random measure on T, then we define *the stochastic integral* $\int_T f\,dM$ of f relative to M by

$$\int_T f\,dM = \sum_{j=1}^{n} a_j M(A_j). \tag{4.6}$$

THEOREM 4.2. *Let* $0 < r < 1$ and $0 < \alpha < 2$. *Let M be an r-SS(α) random measure on $(T, \mathscr{B}(T))$ with control measure m (if M is nonsymmetric α is assumed $\neq 1$). Then, for every $0 < q < \alpha$, there exists a linear map ψ: $L_\alpha(T, |s|^\alpha\,dm) \to L_q(\Omega, P)$ such that $\psi(f) = \int_T f\,dM$ (the stochastic integral of f relative to M) is an r-SS(α)* r.v., *which is symmetric if M is symmetric, and has the* ch. *function*

$$\hat{\mathscr{L}}^{(t)}_{\int_T f\,dM} = \exp\left\{ -\int_T |tsf(s)|^\alpha k_\alpha(tsf(s))\,dm \right\}, \tag{4.7}$$

if M nonsymmetric; and

$$\hat{\mathscr{L}}^{(t)}_{\int_T f\,dM} = \exp\left\{ -\int_T |tsf(s)|^\alpha \bar{k}_\alpha(tsf(s))\,dm \right\}, \tag{4.8}$$

if M is symmetric. Further, $\psi(f)$ agrees with (4.6) *when f is simple; and for every f in $L_\alpha(T, |x|^\alpha\,dm)$ the following crucial inequality holds:*

$$C_0\left(\int_T |sf(s)|^\alpha\,dm\right)^{q/\alpha} \leqslant E\left|\int_T f\,dM\right|^q \leqslant C_1\left(\int_T |(sf(s)|^\alpha\,dm\right)^{q/\alpha}, \tag{4.9}$$

where C_0 and C_1 depend only on r, α and q (not on f, M, or m) and satisfy $0 < C_0 < C_1 < \infty$. Inequality (4.9) *signifies that the linear map ψ is a topological isomorphism from $L_\alpha(T, |s|^\alpha\,dm)$ onto $\psi(L_\alpha(T, |s|^\alpha\,dm)) \subseteq L_q(\Omega, P)$.*

Proof. We prove the result first when M is nonsymmetric (and $\alpha \neq 1$). Let $f \in L_\alpha(T, |s|^\alpha\,dm)$ be simple; and define $\psi(f) = \int_T f\,dM$, as in (4.6). That Eq. (4.7) holds for such an f, follows from the definition of M and the fact that the joint ch. function of m independent r.v.'s is the product of individual ch. functions; that $\int_T f\,dM$ is r-SS(α) r.v. follows from converse part of Theorem 3.1 and (4.7). We now prove the inequality (4.9) for such an f.

Using Tonelli's Theorem, one notes that for any r.v. ξ and $0<p<2$,

$$E|\xi|^p = C\int_0^\infty \frac{E(1-\cos 2s\xi)}{s^{1+p}}\,ds, \qquad \text{where } C^{-1} = 2^p\int_0^\infty \frac{1-\cos s}{s^{1+p}}\,ds.$$

Therefore, for the given q,

$$E\left|\int_T f\,dM\right|^q = C\int_0^\infty \left(\frac{1-\mathscr{R}e\phi(2s)}{s^{1+q}}\right)ds, \tag{4.10}$$

where $\phi(t) \equiv \hat{\mathscr{L}}_{\int_T f\,dM}(t)$. Writing $\beta(t)$ for $-\log\phi(t)$, we see that $\mathscr{R}e\phi(t) = \exp\{-\mathscr{R}e\beta(t)\}\cos \mathscr{I}m\beta(t)$; therefore, using the inequalities

$$1 - s^2/2 \leqslant \cos s \leqslant 1, \tag{4.11}$$

we get the bounds

$$1 - e^{-\mathscr{R}e\beta(2t)} \leqslant 1 - \mathscr{R}e\phi(2t) \leqslant 1 - e^{-\mathscr{R}e\beta(2t)}(1 - \tfrac{1}{2}(\mathscr{I}m\beta(2t))^2). \tag{4.12}$$

Now, since clearly

$$\beta(2t) = \int_T |2tsf(s)|^\alpha k_\alpha(2tsf(s))\,dm,$$

it follows from (3.3) and (4.12) that

$$1 - e^{-c_0\int_T |2tsf(s)|^\alpha\,dm} \leqslant 1 - \mathscr{R}e(\phi(2t))$$

and (4.13)

$$1 - \mathscr{R}e(\phi(2t)) \leqslant 1 - e^{c'\int_T |2tsf(s)|^\alpha\,dm}\left\{1 - \frac{c_1^2}{2}\left(\int_T |2tsf(s)|^\alpha\,dm\right)^2\right\},$$

where

$$c' = \begin{cases} c_1 & \text{if} \quad 0 < t \leqslant \dfrac{2^{1/2\alpha}}{c_1^{1/\alpha}(\int_T |2sf(s)|^\alpha\,dm)^{1/\alpha}} \\ c_0 & \text{if} \quad t > \dfrac{2^{1/2\alpha}}{c_1^{1/\alpha}(\int_T |2sf(s)|^\alpha\,dm)^{1/\alpha}}. \end{cases}$$

Using (4.10), (4.13) and the transformation $u = t[\int_T |sf(s)|^\alpha \, dm]^{1/\alpha}$, we get the desired inequality (4.9) by taking $C_0 = C \int_0^\infty (1 - e^{-c_0(2s)^\alpha})/s^{1+q}) \, ds$ and

$$C_1 = C \int_0^d \left\{ \left(1 - e^{-c_1(2s)^\alpha} \left(1 - \frac{c_1(2s)^\alpha}{2} \right) \right) \Big/ s^{1+q} \right\} ds$$
$$+ C \int_d^\infty \left\{ \left(1 - e^{-c_0(2s)^\alpha} \left(1 - \frac{c_1(2s)^\alpha}{2} \right) \right) \Big/ s^{1+q} \right\} ds,$$

where $d = c_1^{-1/\alpha} 2^{1/2\alpha - 1}$.

Now if $f \in L_\alpha(T, |s|^\alpha \, dm)$, one takes a sequence $\{f_n\}$ of simple functions with $f_n \to f$ in $L_\alpha(T, |s|^\alpha \, dm)$. According to (4.9), $\{\int_T f_n \, dM\}$ is Cauchy in $L_q(\Omega, P)$ and hence converges in $L_q(\Omega, P)$ to a random variable which we denote by $\psi(f) \equiv \int_T f \, dM$. Now using (4.7) and (4.9) which have already been proved for simple functions, it is routine to check that $\int_T f \, dM$ is an r-SS(α) r.v. and that (4.7) and (4.9) hold for every $f \in L_\alpha(T, |s|^\alpha \, dm)$. This completes the proof in the nonsymmetric case. The proof in the symmetric case is similar and in fact simpler; the only basic difference in the proof is the derivation of inequality (4.9). Here the inequality analogous to (4.13) is the following:

$$1 - e^{-c_0 \int_T |2tsf(s)|^\alpha \, dm} \leqslant 1 - \phi(2t) \leqslant 1 - e^{-c_1 \int_T |2tsf(s)|^\alpha \, dm}, \tag{4.14}$$

which is obtained directly (without appealing to (4.11)) using (4.8) and bounds c_0 and c_1 of $\bar{k}_\alpha$ from (3.4). Now inequality (4.9) follows using (4.10), (4.14) and the same transformation as before. In this case, the constants C_0 and C_1 are given by

$$C_0 = C \int_0^\infty \left(\frac{1 - e^{-c_0(2s)^\alpha}}{s^{1+q}} \right) ds \quad \text{and} \quad C_1 = C \int_0^\infty \left(\frac{1 - e^{-c_1(2s)^\alpha}}{s^{1+q}} \right) ds.$$

Remark 4.3. (i) The result corresponding to Theorem 4.2, when M is a nonsymmetric $S(\alpha)$ random measure, yields the definition and properties of stochastic integrals relative to such random measures. To state this corresponding result, the only worth noting change to be made in the statement of Theorem 4.2 is that Eq. (4.7) be replaced by

$$\mathscr{L}_{\int_T f \, dM}^{(t)} = \exp\left\{ -|t|^\alpha \int_T |sf(s)|^\alpha \, h_\alpha(tsf(s)) \, dm \right\}; \tag{4.15}$$

and the worth pointing out change to be made in the proof is that in (4.12) one should replace c_0 by $1 \equiv \mathscr{R}(h_\alpha)$ and c_1 by $|\sec \pi\alpha/2| = \sup_{t \in R} |h_\alpha(t)|$.

(ii) In the case when M is symmetric $S(\alpha)$, $0<\alpha<2$, random measure, it follows, from the definition of M, that

$$\phi(t)=\hat{\mathscr{L}}_{\int_T f\,dM}(t)=\exp-\left\{\int_T |tsf(s)|^\alpha\,dm\right\},$$

for every simple function f. Hence, using (4.10) and the same transformation $u=t(\int_T|sf(s)|^\alpha\,dm)^{1/\alpha}$, we obtain $E|\int_T f\,dM|^q=C_2(\int_T|sf(s)|^\alpha\,dm)^{q/\alpha}$, where

$$C_2=C\int_0^\infty\left(\frac{1-e^{-2^\alpha u^\alpha}}{u^{1+q}}\right)du,$$

and C is as in the proof of Theorem 4.3. Therefore, in this case the metric space $L_\alpha(T,|s|^\alpha\,dm)$ is isometrically isomorphic (up to a constant) to $\psi(L_\alpha(T,|s|^\alpha\,dm))$.

The following proposition shows that $L_q(\Omega,P)$ convergence and convergence in probability for r-SS(α) and S(α) r.v.'s are equivalent. Thus, in Theorem 4.2 and the remarks that follow the theorem, we can replace everywhere $L_q(\Omega,P)$-convergence by convergence in probability, if we so desire.

PROPOSITION 4.4. *Let $0<\alpha<2$, $\alpha\neq 1$ and $0<r<1$. Let X, X_n, $n=1,2,\ldots$, be random variables on a probability space (Ω,F,P) such that, for each n, X_n and X have a jointly r-SS(α) distribution. Then, for any $q\in(0,\alpha)$, we have $\{X_n\}$ converges in probability to $X(X_n\to^{\text{pr.}}X)$ if and only if $E|X_n-X|^q\to 0$. If the joint distribution of (X_n,X) is symmetric, then the same result holds for any $\alpha\in(0,2)$, including $\alpha=1$.*

Proof. We will show that $X_n\to^{\text{pr.}}X$ implies $E|X_n-X|^q\to 0$. The converse is trivial and the proof in the symmetric case is similar. Without loss of generality, we may take $X=0$ since, by Theorem 3.1, X_n-X is also r-SS(α) r.v. Further, by Proposition 3.4, we may take X_n to be strictly r-SS(α). Hence, we consider a sequence $\{X_n\}$ of strictly r-SS(α) r.v.'s such that $X_n\to^{\text{pr.}}0$. Now we have

$$\phi_n(t)\equiv Ee^{itX_n}=\exp-\left\{\int_T |ts|^\alpha k_\alpha(ts)\,dm_n\right\},$$

where $T=\{s\in R\colon r^{1/\alpha}<|s|\leqslant 1\}$ and m_n is a finite measure on T. Since $\phi_n(t)\to 1$, we obtain $\int_T|ts|^\alpha k_\alpha(ts)\,dm_n\to 0$. Using (3.4), we obtain $|t|^\alpha r^{2/\alpha}c_0 m_n(T)\to 0$ which implies $m_n(T)\to 0$. Now, using similar arguments as in the proof of Theorem 4.2, we conclude that $E|X_n|^q\leqslant$

const. $(\int_T |s|^\alpha \, dm_n)^{q/\alpha} \leqslant$ const. $[m_n(T)]^{q/\alpha}$. Hence, the result follows, since $m_n(T) \to 0$, as $n \to \infty$.

Remark 4.5. (i) Let $0<r<1$, and $0<\alpha<2$, $\alpha \neq 1$; and let $T=[a, b]$ be a finite interval and m be a measure on T satisfying (4.2) and $m\{a\}=0$. An independent increment strictly r-SS(α) process $\{\xi_\lambda, \lambda \in T\}$ will be called to have control measure m, if $\hat{\mathscr{L}}_{\xi_\lambda}(t)$ is given by the right side (4.3) with A replaced by $[a, \lambda]$. Such processes which are, in addition, right continuous in probability would be denoted here by $\mathscr{N}$. It follows directly (via [19]) or using [13, 14] that there is a natural 1-to-1 correspondence between $\mathscr{N}$ and the class $\mathscr{M}$ of r-SS(α) random measures (on T) with given control measure m. (Note that the condition $m\{a\}=0$ forces that $\xi_a = M\{a\} = 0$). This remark, obviously, applies as well to symmetric r-SS(α) and to both symmetric and nonsymmetric S(α) random measures and processes. We conclude this section by pointing out that Theorem 3.1 and Corollary 3.1 of Schilder [16] follow from our result contained in Remark 4.3(ii). We omit the details.

5. Spectral Representation of Semistable and Stable Processes

Before we discuss the results of this section, we need some more terminology:

Let Λ be any index set and let $0<r<1$, $0<\alpha<2$, $\alpha \neq 1$. Let $X=\{X_\lambda : \lambda \in \Lambda\}$ be an r-SS(α) stochastic process on a probability space (Ω, F, P) and let $\{Y_\lambda\}$ and θ be, respectively, the corresponding centered r-SS(α) process and the centering function. Let $\{\lambda_n\}$ be any countable subset of Λ and let $\{a_n\}$ be a sequence of positive numbers chosen so that $\sum_{n=1}^{\infty} a_n^{-2} Y_{\lambda_n}^2 < \infty$ a.s. Let $Y_n = Y_{\lambda_n}$ and define a map $\Phi: \Omega \to l_2$ by $\Phi(\omega) = \{a_n^{-1} Y_n(\omega)\}$, if $\{a_n^{-1} Y_n(\omega)\} \in l_2$, and $\Phi(\omega)=0$, otherwise. Then, it follows from Proposition 3.4 that $\mu = P \cdot \Phi^{-1}$ is a strictly r-SS(α) p. measure on l_2. Let Γ be the spectral measure on the ring Δ_0 of l_2 corresponding to the measure μ. Let $\tau(s) = \{\psi_n(s)\}$ be a Borel isomorphism of $T \equiv \{s \in R: r^{1/\alpha} < |s| \leqslant 1\}$ onto Δ_0 [11, p. 14]; define $m(A) = \Gamma(\tau(A))$ for each set A in $\mathscr{B}(T)$, the Borel sets of T. Note that m is a finite Borel measure on T. Let $f_n = a_n \psi_n$, $Y_n \equiv Y_{\lambda_n}$, and let $\mathscr{M}$ and $\mathscr{N}$ denote, respectively, the closures of the spans of $\{Y_n\}$ and $\{\psi_n\}$ in the spaces $L_q(\Omega, P) \equiv L_q(\Omega, F, P)$ and $L_\alpha(T, m) \equiv L_\alpha(T, \mathscr{B}(T), m)$ where $0<q<\alpha$. Finally, let M be an r-SS(α) random measure on $(T, \mathscr{B}(T))$ with control measure m.

With the above notation and conventions, we have the following theorem:

THEOREM 5.1. *The map* ψ *from* $\operatorname{sp}\{Y_n\}$ *onto* $\operatorname{sp}\{f_n\}$ *defined by* $\psi(\sum_{j=1}^m b_j Y_{n_j}) = \sum_{j=1}^m b_j f_{n_j}$ *extends to* $\mathcal{M}$ *as a linear topological isomorphism from* $\mathcal{M}$ *onto* $\mathcal{N}$. *Further, for any* n *and any* $\xi_1,\ldots,\xi_n \in \mathcal{M}$, $\mathcal{L}_{(\xi_1,\ldots,\xi_n)}$ *is an* r-SS(α) p. *measure* on R^n *and*

$$\mathcal{L}_{(\xi_1,\ldots,\xi_n)} = \mathcal{L}_{(\eta_1,\ldots,\eta_n)}, \tag{5.1}$$

where $n_j = \int_T s^{-1}\psi(\xi_j)(s)\,dM$, $j = 1,\ldots,n$. *If* $\{X_\lambda\} \subseteq \mathcal{M}' \equiv$ *the closure of the span of* $\{X_{\lambda_n}\}$ *in* $L_q(\Omega, P)$ *(this holds, e.g., if* Λ *is a metric space,* $\{X_\lambda\}$ *is continuous in probability and* $\{\lambda_n\}$ *is dense in* Λ*), then* $\{X_\lambda\}$ *is stochastically equivalent to* $\{\int_T s^{-1}f_\lambda(s)\,dM + \theta(\lambda)\}$ *and* $\{Y_\lambda\}$ *is stochastically equivalent to* $\{\int_T s^{-1}f_\lambda(s)\,dM\}$, *where* $f_\lambda = \psi(Y_\lambda)$.

Proof. First we observe that, using change of variable formula, Theorem 4.2 and the definition of m, we have

$$\mathcal{L}_{(Y_1,\ldots,Y_n)} = \mathcal{L}_{(\int_T s^{-1}f_1(s)\,dM,\ldots,\int_T s^{-1}f_n(s)\,dM)}, \tag{5.2}$$

for all n. The same arguments yield that, for any n and any $\xi_1,\ldots,\xi_n \in \operatorname{sp}\{Y_n\}$, $\mathcal{L}_{(\xi_1,\ldots,\xi_n)} = \mathcal{L}_{(\eta_1,\ldots,\eta_n)}$. Now for every ξ in $\operatorname{sp}\{Y_n\}$ we have, from (4.9),

$$C_0\left(\int_T \left|\frac{\psi(\xi)(s)}{s}\right|^\alpha dm\right)^{q/\alpha} \leqslant E|\xi|^q \leqslant C_1\left(\int_T \left|\frac{\psi(\xi)(s)}{s}\right|^\alpha dm\right)^{q/\alpha};$$

this immediately shows that ψ extends to $\mathcal{M}$ as a linear topological isomorphism onto $\mathcal{N}$. Now, let $\xi_1,\ldots,\xi_n \in \mathcal{M}$, $g_j = \psi(\xi_j)$, and $n_j = \int_T s^{-1}g_j(s)\,dM$, $j = 1,\ldots,n$. Let $t_1,\ldots,t_n \in R$; then using the fact that ψ is a topographical isomorphism and the usual limiting argument one can show that $\hat{\mathcal{L}}_{(\xi_1,\ldots,\xi_n)}(t_1,\ldots,t_n) = \hat{\mathcal{L}}_{(\eta_1,\ldots,\eta_n)}(t_1,\ldots,t_n)$; proving (5.1). Now the only thing that remains to be shown is the last part. We outline this below:

First, we note that if $\{X_\lambda\} \subseteq \mathcal{M}'$ then $\{Y_\lambda\} \subseteq \mathcal{M}$. For, if for a fixed λ, X_λ is the $L_q(\Omega, P)$ limit of $\{\xi_j'\} \subseteq \operatorname{sp}\{X_{\lambda_n}\}$, then $\xi_j' = \eta_j' + z_j$, with $\eta_j' \in \operatorname{sp}\{Y_n\}$. It follows, from Proposition 3.3, that $z_j \to \theta(\lambda)$ and $\eta_j' \to Y_\lambda$ in $L_q(\Omega, P)$. Hence $Y_\lambda \in \mathcal{M}$. Since we have already shown that $\{Y_\lambda\}$ is stochastically equivalent to $\{\int_T s^{-1}\psi(Y_\lambda)(s)\,dM\}$ and $X_\lambda = Y_\lambda + \theta(\lambda)$, it follows that $\{X_\lambda\}$ is stochastically equivalent to $\{\int_T s^{-1}f_\lambda(s)\,dM + \theta(\lambda)\}$.

Variations of Theorem 5.1 for symmetric r-SS(α) processes and symmetric and nonsymmetric $S(\alpha)$ processes can be obtained by making obvious changes in the statement and in the proof of Theorem 5.1. (We note that in the symmetric case, for both r-SS(α) and S(α) processes, the representation is valid for all $\alpha \in (0, 2)$.) Since all these changes are straightforward, we omit the details. But some clarification in the non-symmetric S(α) case ($\alpha \neq 1$) is needed: First, we note that since on a separable

Banach a p. measure μ is $S(\alpha) \Leftrightarrow$ it is an $r-\mathrm{SS}(\alpha)$ for every $0<r<1$ (see [4]), both Propositions 3.3 and 3.4 hold for the $\mathrm{S}(\alpha)$ case. In view of this and the proof of Theorem 5.1, one only needs to verify as to why Eq. (5.2) holds when the process $Y=\{Y_\lambda : \lambda \in \Lambda\}$ is a centered $\mathrm{S}(\alpha)$ process. (See the notation prior to Theorem 5.1; we must emphasize that Δ_0 in $\mathrm{S}(\alpha)$ case is replaced by $\Delta=\{x \in l_2 : \|x\|=1\}$, and T can be taken $[0, 1]$, see Remark 5.2). Now to see the validity of (5.2), we observe

$$\hat{\mathscr{L}}_{(Y_1, \ldots, Y_n)}(t_1, \ldots, t_n)=\exp-\left\{\int_\Delta \left|\sum_{j=1}^n a_j t_j x_j\right|^\alpha h_\alpha\left(\sum_{j=1}^n a_j t_j x_j\right) d\Gamma\right\},$$

where h_α is as in (4.5) and Γ is the spectral measure on Δ of the $\mathrm{S}(\alpha)$ p. measure μ on l_2 (see discussion prior to Theorem 5.1); also

$$\hat{\mathscr{L}}^{(t_1, \ldots, t_n)}_{(\int_T s^{-1} f_1(s)\, dM, \ldots, \int_T s^{-1} f_n(s)\, dM)} = \hat{\mathscr{L}}^{(1)}_{\int_T \{s^{-1} \sum_{j=1}^n t_j f_j(s)\}\, dM}$$

$$= \exp - \left\{\int_T \left| s \cdot s^{-1} \sum_{j=1}^n t_j f_j(s)\right|^\alpha h_\alpha\left(s \cdot s^{-1} \sum_{j=1}^n t_j f_j(s)\right) dm\right\}$$

(by (4.15))

$$= \exp - \left\{\int_T \left|\sum_{j=1}^n t_j f_j(s)\right|^\alpha h_\alpha\left(\sum_{j=1}^n t_j f_j(s)\right) dm\right\}.$$

We point out that the ideas of embedding $\{a_n^{-1} Y_n\}$ in l_2 and defining the random measure M discussed prior to Theorem 5.1 are similar to those used by Kuelbs [7] and Schilder [16].

Remark 5.2. (i) We note that in Theorem 5.1 the fact that the set T is of the form $(r^{1/\alpha}<|x| \leqslant 1)$ is not essential, and it can be replaced by any uncountable Borel set as long as the measure m on T satisfies the condition $\int_T |x|^\alpha\, dm<\infty$.

(ii) In the case when $\{\xi_\lambda : \lambda \in \Lambda\}$ is a symmetric $\mathrm{S}(\alpha)$ process, we can define suitably a random measure M' on any finite given interval $T=[a, b]$ so that $\{\int_T f_\lambda\, dM'\}$ (rather than $\{\int_T f_\lambda(x)/x\, dM'\}$) is stochastically equivalent to $\{\xi_\lambda : \lambda \in \Lambda\}$. To see this, let $m=\Gamma \circ \tau$ be as in the theorem, where now τ is a Borel isomorphism of Δ_0 onto $[a, b]$ and let $dm'=(dm/|x|^\alpha)$. Then, if M' is a symmetric $\mathrm{S}(\alpha)$ random measure with control measure m', then it follows, as in the proof of the theorem, that $\{\xi_\lambda : \lambda \in \Lambda\}$ is stochastically equivalent to $\{\int_T f_\lambda\, dM'\}$ (this, via Remark in the end of Section 4, agrees with the representation obtained by Kuelbs [7] and Schilder [16]).

(iii) Our approach of defining r. measures and stochastic integrals has two drawbacks: First the distribution of $M(A)$ is not invariant under translation of A when the control measure m of the r. measure M is the Lebesgue measure. Second, the multipliers s^{-1} need to be inserted in the representations of r-SS(α) and S(α) processes (see Theorem 5.1). Both of these drawbacks arise because we require $\hat{\mathscr{L}}_{M(A)}$ satisfy (4.3) (and (4.4) in S(α) case); if we have, instead, required

$$\hat{\mathscr{L}}_{M(A)}(t) = \exp - \left\{ \lambda(A) \int_{\Delta_0} |ts|^{\alpha} k_{\alpha}(ts)\, \Gamma(ds) \right\},$$

where λ is a measure on T and Γ a finite measure on Δ_0, then both of these drawbacks can be overcome. This point has been considered in details in our forthcoming paper [15], where we consider the spectral representation of complex r-SS(α) and other infinitely diversible processes. Our initial approach in the present paper was mainly motivated by two considerations: First, we wanted to avoid the dependence of $\hat{\mathscr{L}}_{M(A)}$ on the product measure $\lambda \times \Gamma$ on $T \times \Delta_0$ and wanted it to depend only on a single measure like m (as in (4.3)) on T; this approach made the proof of the representation Theorem 5.1 somewhat easier. Second, we wanted our analysis as close as possible to that of symmetric S(α) case of [7] where $\hat{\mathscr{L}}_{M(A)}$ depends only on a measure on T.

6. An Independence Criterion for I. D. Random Variables

In this section, we obtain a criterion of independence for Banach valued i.d. r.v.'s and real stochastic integrals. We begin with the statement of the main theorem.

Theorem 6.1. *Let B_j, $j = 1, \ldots, n$, be n-separable Banach spaces and let X_j be a B_j-valued* r.v., *$j = 1, \ldots, n$. Let $\mu = \mathscr{L}_X$ be the law of $X = (X_1, \ldots, X_n)$ on the Banach space $B = B_1 \times \cdots \times B_n$; and assume that μ is Poisson-type* i.d. *measure with Lévy measure F. Then the* r.v.'s *$X_1, \ldots, X_n$ are independent if and only if*

$$F\left(\bigcup_{j=1}^{n} I_j \right)^c = 0, \tag{6.1}$$

where $I_j = \{x \in B\colon x_k = 0,\ k \neq j,\ k = 1, \ldots, n\}$ (i.e., F is concentrated on the "coordinate axes").

Proof. Let (6.1) hold; and let π_j be the natural projection of B onto I_j.

Then, using the form of the ch. function of Poisson type i.d. measures on Banach spaces (see, e.g., [1]), it is easy to verify that

$$\hat{\mathscr{L}}_X(y_1,..., y_n) = \prod_{j=1}^{n} \hat{\mathscr{L}}_{\pi_j(X)}^{(y_1,..., y_n)};$$

for every $y_j \in B_j^*$, $j = 1,..., n$, and hence $X_1,..., X_n$ are independent.

To prove the converse, first we note that the symmetrized p. measure $\mu * \bar{\mu}$ is i.d. with Lévy measure $F + \bar{F}$. Since $(F + \bar{F})\,(\bigcup_{j=1}^{n} I_j)^c = 0$ implies $F(\bigcup_{j=1}^{n} I_j)^c = 0$, it is sufficient to prove the result assuming that both μ and F symmetric. We define a Borel measure G on B as the restriction to $B\backslash\{0\}$ of the measure $\sum_{j=1}^{n} F \circ \pi_j^{-1}(A \cap I_j)$, where A is any Borel set of B. Then, using the definition of G, we have

$$\int_{I_j} I_{A_j}(\pi_j(x))\, dG = \int_B I_{A_j}(\pi_j(x))\, dF.$$

This, along with usual limit arguments show that, for $y_j \in B_j^*$, $j = 1,..., n$,

$$\int_{I_j} \psi(\langle \pi_j(x), \tilde{y}_j \rangle)\, dG = \int_B \psi(\langle \pi_j(x), \tilde{y}_j \rangle)\, dF, \tag{6.2}$$

where $\psi(\lambda) = \cos \lambda - 1$, and $\tilde{y}_j \in B^*$ is defined by $\tilde{y}_j(x_1,..., x_n) = y_j(x_j)$. By independence of X_j's and (6.2) we have

$$\begin{aligned}
\hat{\mu}(y_1,..., y_n) &= \prod_{j=1}^{n} \int_B e^{i\langle \pi_j(x), \tilde{y}_j \rangle d\mu} \\
&= \prod_{j=1}^{n} \exp\left\{\int_B \psi(\langle \pi_j(x), \tilde{y}_j \rangle)\, dF\right\} \\
&= \exp\left\{\sum_{j=1}^{n} \int_{I_j} \psi(\langle \pi_j(x), \tilde{y}_j \rangle)\, dG\right\} \\
&= \exp\left\{\sum_{j=1}^{n} \int_{I_j} \psi\left(\sum_{k=1}^{n} \langle \pi_k(x), \tilde{y}_k \rangle\right) dG\right\} \\
&= \exp\left\{\int_B \psi\left(\sum_{k=1}^{n} \langle \pi_k(x), \tilde{y}_k \rangle\right) dG\right\};
\end{aligned}$$

therefore

$$\exp\left\{\int_B \psi\left(\sum_{k=1}^{n} \langle \pi_k(x), \tilde{y}_k \rangle\, dF\right\} = \exp\left\{\int_B \psi\left(\sum_{k=1}^{n} \langle \pi_k(x), \tilde{y}_k \rangle\right) dG\right\}.$$

Hence, applying the arguments used to prove the uniqueness for Lévy measures (see, e.g., [1, p. 56]), we conclude $F=G$. Thus, since $G(\bigcup_{j=1}^{n} I_j)^c=0$, the proof of the converse part of the theorem follows.

Remark 6.2. We point out that the proof of the above theorem for real valued i.d. r.v.'s attempted in [12] is erroneous because the author used an incorrect inequality. Since r-SS(α) and S(α) Banach valued r.v.'s are i.d., Theorem 6.1 applies to such r.v.'s. Further, in these cases, the condition (6.1) can be expressed in terms of the spectral measures Γ of $\mathscr{L}_{(X_1,\ldots,X_n)}$. In fact, in the semistable case, condition (6.1) is equivalent to $\Gamma(\Delta_0 \cap (\bigcup_{j=1}^{n} I_j)^c)=0$, since $\Gamma=F/\Delta_0$; and, in the stable case (6.1) is equivalent to $\Gamma(\Delta \cap (\bigcup_{j=1}^{n} I_j)^c)=0$.

This follows from the well-known relations

$$\Gamma(A)=\alpha F\left\{x\in B\colon \|x\|\geqslant 1, \frac{x}{\|x\|}\in A\right\}$$

and

$$F(A)=\int_{\Delta}\int_{0}^{\infty} I_A(r\cdot x)\, r^{-1-\alpha} dr\, d\Gamma,$$

where A is any Borel set of B and α is the index of μ (see, e.g., [1, p 149]). This last fact, for real symmetric r.v.'s, have been obtained earlier by Miller [10].

Theorem 6.1 also yields the following important criterion of independence for stochastic integrals relative to semistable and stable random measures; this extends a result of Schilder [16]. The simple proof is omitted.

COROLLARY 6.3. *Let M be an r-SS(α) (or S(α)) independently scattered random measure with control measure m defined on a Borel subset T of R (if M is nonsymmetric, α is assumed $\neq 1$). Let $f_1,\ldots,f_n$ belong to $L_\alpha(T, |x|^\alpha\, dm)$, then the* r.v's $\int_T f_1\, dM,\ldots, \int_T f_n\, dM$ *are independent if and only if*

$$m\left\{s\colon \sum_{i\neq j} |f_i(s) f_j(s)| \neq 0\right\}=0$$

(*i.e.*, f_j's *have "disjoint support"*).

Remark 6.4. This corollary can also be obtained by a general criterion of independence of stochastic integrals obtained by Urbanik [18].

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